Two Dimensional Cantilever Beam Analysis

Efficiency Comparison: Elasticity, Mechanics of Materials, Finite Elements Analysis

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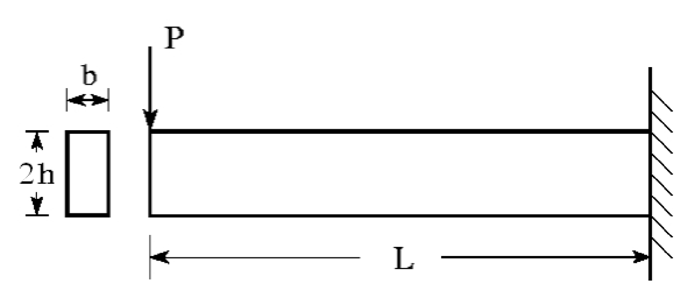
Abstract:

Modern structural mechanics utilizes three approaches to determining the affects a load has on an elastic body:

- Mechanics of Materials Methods
- Elasticity Theory
- Finite Elements Analysis

Each of the above mention analysis techniques have specific characteristics that make them desirable analysis methods, both in engineering academia and industry. This presentation will demonstrate some key principles of the above analysis methods in order to analyze a typical structural engineering scenario, the cantilever beam.

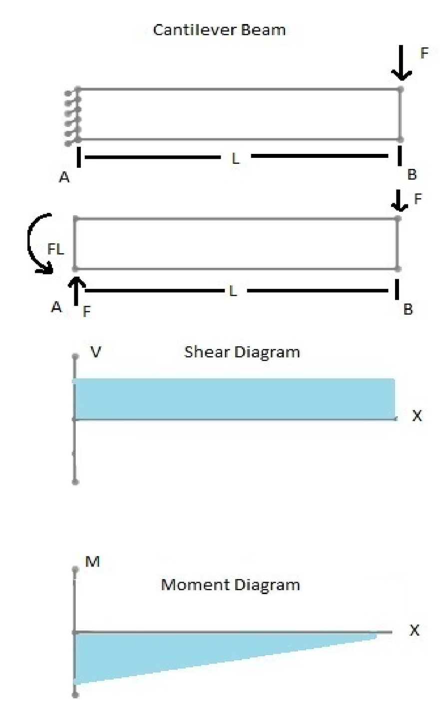
Problem statement:



Calculate the expressions for **Stress**, **Strain**, and **Displacement** of the above loading situation, using both the Elasticity and Mechanics of Materials analysis methods.

Mechanics of Materials Methods:

The first step to approaching a mechanics of materials problem is to identify the loads and to solve for the reactions of the system. In this case the results were produced using shear and bending moment diagrams, to give a graphical representation.



Now that we have the reaction forces for this particular case we can use them to solve the bending stress induced and in-turn the strain in the system. The bending stress in the beam can be seen in the diagram below. The magnitude of the stress is governed by the following equation:

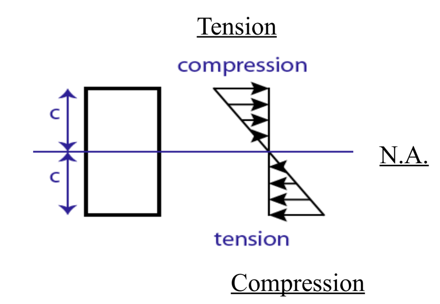
$$\sigma_b = \frac{My}{I}$$

Where:

M is the calculated moment from the bending diagram.

Y is the distance from the neutral axis to the point at which the bending stress is to be evaluated.

I is the moment of inertia. This is a geometry dependent property.



Once the bending stress has been solved for, the next step is to calculate the strain at that particular stress point. Tis is easily done by solving the following equation:

Where: $\sigma = \varepsilon E$

E is the Modulus of Elasticity, and is material dependent.

Elasticity Method:

The first step to approaching an Elasticity problem is to identify the fundamental conditions surround the problem. These conditions are defined by certain physical laws and material properties.

- The Equations of Equilibrium must be satisfied.

$$\left(\frac{\partial \sigma_x}{\partial x}\right)+\left(\frac{\partial \tau_{xy}}{\partial y}\right)+F_x=0 \qquad \left(\frac{\partial \sigma_y}{\partial y}\right)+\left(\frac{\partial \tau_{xy}}{\partial x}\right)+F_y=0$$

- Hooke's Law must apply to the given material.
- The “Compatibility Equation” due to the boundary conditions.

$$\left(\left(\frac{\partial^2}{\partial x^2}\right)+\left(\frac{\partial^2}{\partial y^2}\right)\right)\left(\sigma_x+\sigma_y\right)=0$$

The second step would be identifying which equations would be used to describe the following problem statement.

Stress:

$$\sigma_x=\frac{\partial^2 \varphi}{\partial y^2}$$

$$\sigma_y=\frac{\partial^2 \varphi}{\partial x^2}$$

$$\tau_{xy}=\frac{-\partial^2 \varphi}{\partial x \partial y}$$

Strain:

$$\varepsilon_x=\frac{1-v^2}{E}*\left(\sigma_x-\frac{v}{1-v}\sigma_y\right)$$

$$\varepsilon_y=\frac{1-v^2}{E}*\left(\sigma_y-\frac{v}{1-v}\sigma_x\right)$$

$$\gamma_{xy}=\frac{\tau_{xy}}{G}$$

Substituting σ(x) and σ(y) in to τ(xy) yields a function know as the **Biharmonic Equation.**

$$\frac{\partial^4 \varphi}{\partial x^4}+2\left(\frac{\partial^4 \varphi}{\partial x^2 \partial y^2}\right)+\frac{\partial^4 \varphi}{\partial y^4}=0 \longrightarrow \nabla^4 \varphi=0$$

The significance of this equation is that we have now condensed the multiple stress and strain equations into this equation which must be satisfied in order to compute the stress and strain. Developing a function for in which satisfied the Biharmonic Equation is done by applying a technique called the **semi-inverse method**.

$$\varphi=a x^4+b x^3 y+c x^2 y^2+dx y^3+e y^4+f x^3+g x^2 y+h' x y^2+i x^3+j x^2 y+k xy+\ldots$$

After solving the Biharmonic, its time to establish the boundary conditions for the set situation.

Top:

$$\sigma_{yy}(x,-h)=0 \qquad 0<x<L$$

$$\tau_{xy}(x,-h)=0 \qquad 0<x<L$$

Bottom:

$$\sigma_{yy}(x,h)=0 \qquad 0<x<L$$

$$\tau_{xy}(x,h)=0 \qquad 0<x<L$$

Right:

$$P_x=\int_{-h}^{h} \sigma_{xx}(L,y)\, t\, dy=0$$

$$P_y=\int_{-h}^{h} \tau_{xy}(L,y)\, t\, dy=-P$$

$$M=\int_{-h}^{h} \sigma_{xx}(L,y)\, y\, t\, dy=-PL$$

Right:

$$P_x=\int_{-h}^{h} \sigma_{xx}(0,y)\, t\, dy=0$$

$$P_y=\int_{-h}^{h} \tau_{xy}(0,y)\, t\, dy=-P$$

$$M=\int_{-h}^{h} \sigma_{xx}(0,y)\, yt\, dy=0$$

Apply boundary conditions and solving yields:

$$\varepsilon_x=\frac{1}{E}\left(\frac{-P}{I}xy-v(0)\right) \qquad \varepsilon_y=\frac{1}{E}\left((0)-v\left(\frac{-P}{I}xy\right)\right) \qquad \gamma_{xy}=\frac{\frac{P}{2I}\left(y^2-h^2\right)}{G}$$

Conclusions:

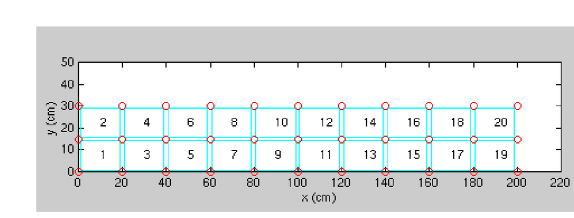
The cantilever beam is a typical scenario in the structural engineering field and being able to solve the system efficiently is crucial. The Elasticity Theory is one of the fundamental methods of calculating the stress and strain in structural mechanics problems. This method entails more tedious calculations versus the more often used Mechanics of Materials Method. However the purpose to the cumbersome derivations leads to a more exact approximation of the stress and strain fields. Increased accuracy in real world situations leads to savings of material cost of the structure. The Elasticity Method is often applied in the aerospace engineering industry, due to the increased accuracy and the tolerance requirements of the systems undertaken. What the Mechanics of Materials method lacks in accuracy is made up in ease/speed of computation. Most building and bridge based structural design is based off of Mechanic of Materials methods. Design of these structures incorporates “safety factors” that either increase the loads or decreased the capacities. These safety factors are incorporated into the design of these systems in order to assure against failure (catastrophic and service); therefore accurate in the calculation can be relaxed to increase ease/speed of calculations. The newest analysis method discussed, the Finite Element Method (FEA), has the advantage of both speed and accuracy (depending on the system). However, when using the FEA method it is important to include the cost of computer computations in the efficiency of the analysis selection. In some systems it is necessary to have thousands or even millions of node points/mesh systems. This requires fairly significant computational power. Another important consideration of the FEA method is the lack of translucence in the output calculations. It is difficult for beginning users to establish a meter for whether or not the output is a reasonable one, due to the amount of input requirements and the amount of places to affect the output calculations. It is important to have a skilled FEA producer when creating the system models.

In conclusion, choosing an analysis process is determined by several major key factors:

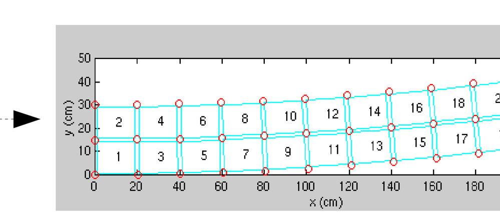
- System Complexity
- Calculation Accuracy Needed
- Available Computational Power
- The Users Mathematical/Computational Ability and Expertise

Finite Element Analysis:

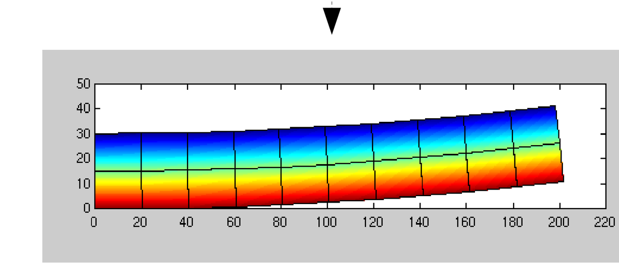
The Finite Element Analysis method is a process in which a solid is divided into multiple sections or elements, then reconnects them at node positions. Its the equivalent to cutting a piece of fruit in pieces then reconnecting them using push pins. These nodes can then be related use a set of simultaneous equations. In this case a software called MATLAB was used to create the model and then segment the model into twenty “elements” and thirty-three nodes. To aid the finite element analysis of this system a plug-in and input code, called Alladin 2.0, created by the University of Maryland's School of Engineering was utilized.



Load The System



Stress Contours



Using MATLAB, a code was written to then enact a load at a nodal position, to simulate a load on the cantilever beam. Since the stiffness of the system is known, once the displacement of each node is calculated, the force and the stress in the system can be calculated.

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